Matrix inequalities including Furuta inequality via Riemannian mean of n-matrices (Structural study of operators via spectra or numerical ranges)

AUTHOR(S):

Ito, Masatoshi

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Matrix inequalities including Furuta inequality via Riemannian mean of n-matrices

前橋工科大学　伊藤 公智 (Masatoshi Ito)
Maebashi Institute of Technology

Abstract

In this report, we shall obtain a generalization of Furuta inequality via weighted Riemannian mean, a kind of geometric mean, of n-matrices. This result is related to Yamazaki's recent results which is a kind of generalizations of Ando-Hiai inequality and Furuta inequality for chaotic order.

1 Introduction

The weighted geometric mean of two positive definite matrices A and B defined by $A \sharp_{\alpha} B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\alpha}A^{\frac{1}{2}}$ for $\alpha \in [0,1]$. In particular, we call $A \sharp_{\frac{1}{2}} B$ (denoted by $A \sharp B$ simply) the geometric mean of A and B. The weighted geometric mean sometimes appears in famous matrix inequalities, for example, Furuta inequality [10] (see also [6, 11, 13, 17, 20]) and Ando-Hiai inequality [1]. We remark that these inequalities hold even in the case of bounded linear operators on a complex Hilbert space. In what follows, we denote $A \geq 0$ if A is a positive semidefinite matrix (or operator), and we denote $A>0$ if A is a positive definite matrix (or operator).

Theorem 1.A (Satellite form of Furuta inequality [10, 17]).

$$A \geq B \geq 0 \textit{ with } A>0 \quad \textit{implies} \quad A^{-r} \sharp_{\frac{1+r}{p+r}} B^{p} \leq B \leq A \quad \textit{for } p \geq 1 \textit{ and } r \geq 0.$$

Theorem 1.B (Ando-Hiai inequality [1]). *For $A, B>0$,*

$$A \sharp_{\alpha} B \leq I \quad \textit{for } \alpha \in (0,1) \quad \textit{implies} \quad A^{r} \sharp_{\alpha} B^{r} \leq I \quad \textit{for } r \geq 1.$$

For $A, B>0$, it is well known that chaotic order $\log A \geq \log B$ is weaker than usual order $A \geq B$ since $\log t$ is a matrix (or operator) monotone function. The following result is known as the Furuta inequality for chaotic order.

Theorem 1.C (Furuta inequality for chaotic order [7, 12]). *Let $A, B>0$. Then the following assertions are mutually equivalent;*

(i) $\log A \geq \log B$,

(ii) $A^{-p} \natural B^p \leq I$ *for all* $p \geq 0$,

(iii) $A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$ *for all* $p \geq 0$ *and* $r \geq 0$.

It has been a longstanding problem to extend the (weighted) geometric mean for three or more positive definite matrices. Many authors attempt to find a natural extension, for example, Ando-Li-Mathias' mean and its refinement [2, 5, 15, 16] and Riemannian mean (or the least squares mean) [4, 18, 19]. We remark that Ando-Li-Mathias [2] originally proposed the following ten properties (P1)–(P10) which should be required for a reasonable geometric mean $\mathfrak{G}$ of positive definite matrices. We note that, in [2], they require continuity from above as (P5).

Let $P_m(\mathbb{C})$ be the set of $m \times m$ positive definite matrices on $\mathbb{C}$. Let $A_i, A_i', B_i \in P_m(\mathbb{C})$ for $i = 1, \ldots, n$ and let $\omega = (w_1, \ldots, w_n)$ be a probability vector. Then

(P1) Consistency with scalars. If $A_1, \ldots, A_n$ commute with each other, then

$$\mathfrak{G}(\omega; A_1, \ldots, A_n) = A_1^{w_1} \ldots A_n^{w_n}.$$

(P2) Joint homogeneity. For positive numbers $a_i > 0$ $(i = 1, \ldots n)$,

$$\mathfrak{G}(\omega; a_1 A_1, \ldots, a_n A_n) = a_1^{w_1} \ldots a_n^{w_n} \mathfrak{G}(\omega; A_1, \ldots, A_n).$$

(P3) Permutation invariance. For any permutation π on $\{1, \ldots n\}$,

$$\mathfrak{G}(\omega; A_1, \ldots, A_n) = \mathfrak{G}(\pi(\omega); A_{\pi(1)}, \ldots, A_{\pi(n)}),$$

where $\pi(\omega) = (w_{\pi(1)}, \cdots, w_{\pi(n)})$.

(P4) Monotonicity. If $B_i \leq A_i$ for each $i = 1, \ldots n$, then

$$\mathfrak{G}(\omega; B_1, \ldots, B_n) \leq \mathfrak{G}(\omega; A_1, \ldots, A_n).$$

(P5) Continuity. For each $i = 1, \ldots n$, let $\{A_i^{(k)}\}_{k=1}^{\infty}$ be positive definite matrix sequences such that $A_i^{(k)} \to A_i$ as $k \to \infty$. Then

$$\mathfrak{G}(\omega; A_1^{(k)}, \ldots, A_n^{(k)}) \to \mathfrak{G}(\omega; A_1, \ldots, A_n) \quad \text{as } k \to \infty.$$

(P6) Congruence invariance. For any invertible matrix S,

$$\mathfrak{G}(\omega; S^* A_1 S, \ldots, S^* A_n S) = S^* \mathfrak{G}(\omega; A_1, \ldots, A_n) S.$$

(P7) Joint concavity.

$$\begin{aligned}&\mathfrak{G}(\omega; \lambda A_1 + (1-\lambda)A_1', \ldots, \lambda A_n + (1-\lambda)A_n') \\ &\geq \lambda \mathfrak{G}(\omega; A_1, \ldots, A_n) + (1-\lambda)\mathfrak{G}(\omega; A_1', \ldots, A_n') \quad \text{for } 0 \leq \lambda \leq 1.\end{aligned}$$

(P8) Self-duality. $\mathfrak{G}(\omega; A_1^{-1}, \ldots, A_n^{-1})^{-1} = \mathfrak{G}(\omega; A_1, \ldots, A_n)$.

(P9) Determinant identity. $\det \mathfrak{G}(\omega; A_1, \ldots, A_n) = \prod_{i=1}^{n} (\det A_i)^{w_i}$.

(P10) The arithmetic-geometric-harmonic mean inequality.

$$\left(\sum_{i=1}^{n} w_i A_i^{-1}\right)^{-1} \leq \mathfrak{G}(\omega; A_1, \ldots, A_n) \leq \sum_{i=1}^{n} w_i A_i.$$

For $A, B \in P_m(\mathbb{C})$, Riemannian metric between A and B is defined as $\delta_2(A, B) = \| \log A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \|_2$, where $\|X\|_2 = (\operatorname{tr} X^* X)^{\frac{1}{2}}$ (details are in [3]). By using Riemannian metric, Riemannian mean is defined as follows:

Definition 1 ([3, 4, 18, 19]). *Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. Then weighted Riemannian mean $\mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \in P_m(\mathbb{C})$ is defined by*

$$\mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) = \underset{X \in P_m(\mathbb{C})}{\arg\min} \sum_{i=1}^{n} w_i \delta_2^2(A_i, X),$$

where $\arg\min f(X)$ *means the point X_0 which attains minimum value of the function $f(X)$. In particular, we call $\mathfrak{G}_\delta(\omega; A_1, \ldots, A_n)$ (denoted by $\mathfrak{G}_\delta(A_1, \ldots, A_n)$ simply) Riemannian mean if $\omega = (\frac{1}{n}, \ldots, \frac{1}{n})$.*

We remark that $\mathfrak{G}_\delta(\omega; A, B) = A \sharp_\alpha B$ for $\alpha \in [0, 1]$ and $\omega = (1-\alpha, \alpha)$ since the property $\delta_2(A, A \sharp_\alpha B) = \alpha \delta_2(A, B)$ holds.

It is shown in [3, 4, 18, 19] that weighted Riemannian mean satisfies (P1)–(P10) (see also [21]). We remark that Riemannian mean has a stronger property (P5') than (P5).

(P5') Non-expansive.

$$\delta_2(\mathfrak{G}_\delta(\omega; A_1, \ldots, A_n), \mathfrak{G}_\delta(\omega; B_1, \ldots, B_n)) \leq \sum_{i=1}^{n} w_i \delta_2(A_i, B_i).$$

Very recently, Yamazaki [21] has obtained an excellent generalization of Theorems 1.B and 1.C via weighted Riemannian mean $\mathfrak{G}_\delta$ of n-matrices. We recall that $\omega = (w_1, \ldots, w_n)$ is a probability vector if the components satisfy $\sum_i w_i = 1$ and $w_i > 0$ for $i = 1, \ldots, n$.

Theorem 1.D ([21]). *Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. Then*

$$\mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \leq I \quad \textit{implies} \quad \mathfrak{G}_\delta(\omega; A_1^p, \ldots, A_n^p) \leq I \ \textit{ for } p \geq 1.$$

Theorem 1.E ([21]). *Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$. Then the following assertions are mutually equivalent;*

(i) $\log A_1 + \cdots + \log A_n \leq 0$,

(ii) $\mathfrak{G}_\delta(A_1^p, \ldots, A_n^p) \leq I$ *for all* $p > 0$,

(iii) $\mathfrak{G}_\delta(\omega; A_1^{p_1}, \ldots, A_n^{p_n}) \leq I$ *for all* $p_1, \ldots, p_n > 0$, *where* $p_{\neq i} = \prod_{j \neq i} p_j$ *and* $\omega = \left(\frac{p_{\neq 1}}{\sum_i p_{\neq i}}, \ldots, \frac{p_{\neq n}}{\sum_i p_{\neq i}} \right)$.

Theorems 1.D and 1.E imply Theorems 1.B and 1.C, respectively, since $\mathfrak{G}_\delta(\omega; A, B) = A \,\natural_\alpha\, B$ for $\omega = (1-\alpha, \alpha)$. Moreover, it has been shown in [21] that Theorem 1.D does not hold for other geometric means satisfying (P1)–(P10).

In this report, corresponding to Theorem 1.E, we shall obtain a generalization of Furuta inequality (Theorem 1.A) via weighted Riemannian mean of n-matrices. Moreover we shall show an extension of Theorem 1.D.

2 Results

Firstly, we show an extension of Theorem 1.D. Theorem 1.D follows from Theorem 2.1 by putting $p_1 = \cdots = p_n = p$.

Theorem 2.1. *Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. If $\mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \leq I$, then*

$$\mathfrak{G}_\delta(\omega'; A_1^{p_1}, \ldots, A_n^{p_n}) \leq \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \leq I \quad \textit{for } p_1, \ldots, p_n \geq 1,$$

where $\widehat{\omega'} = (\frac{w_1}{p_1}, \ldots, \frac{w_n}{p_n})$ and $\omega' = \frac{\widehat{\omega'}}{\|\widehat{\omega'}\|_1}$.

We remark that $\|\cdot\|_1$ means 1-norm, that is, $\|x\|_1 = \sum_i |x_i|$ for $x = (x_1, \ldots, x_n)$. In order to prove Theorem 2.1, we use the following results.

Theorem 2.A ([18, 19]). *Let $A_1,\dots,A_n \in P_m(\mathbb{C})$ and $\omega=(w_1,\dots,w_n)$ be a probability vector. Then $X=\mathfrak{G}_\delta(\omega;A_1,\dots,A_n)$ is the unique positive solution of the following matrix equation:*

$$w_1 \log X^{\frac{-1}{2}}A_1X^{\frac{-1}{2}}+\cdots+w_n\log X^{\frac{-1}{2}}A_nX^{\frac{-1}{2}}=0.$$

Theorem 2.B ([21]). *Let $A_1,\dots,A_n \in P_m(\mathbb{C})$ and $\omega=(w_1,\dots,w_n)$ be a probability vector. Then*

$$w_1\log A_1+\cdots+w_n\log A_n\le 0 \quad \textit{implies} \quad \mathfrak{G}_\delta(\omega;A_1,\dots,A_n)\le I.$$

Proof of Theorem 2.1. Let $X=\mathfrak{G}_\delta(\omega;A_1,\dots,A_n)\le I$. Then for each $p_1,\dots,p_n\in[1,2]$, by Theorem 2.A and Hansen's inequality [14],

$$\begin{aligned} 0 &= \frac{1}{\|\widehat{\omega'}\|_1}\sum w_i\log X^{\frac{1}{2}}A_i^{-1}X^{\frac{1}{2}} = \frac{1}{\|\widehat{\omega'}\|_1}\sum \frac{w_i}{p_i}\log(X^{\frac{1}{2}}A_i^{-1}X^{\frac{1}{2}})^{p_i} \\ &\le \frac{1}{\|\widehat{\omega'}\|_1}\sum \frac{w_i}{p_i}\log X^{\frac{1}{2}}A_i^{-p_i}X^{\frac{1}{2}}, \end{aligned}$$

that is, $\sum \frac{\frac{w_i}{p_i}}{\|\widehat{\omega'}\|_1}\log X^{\frac{-1}{2}}A_i^{p_i}X^{\frac{-1}{2}}\le 0$. By applying Theorem 2.B,

$$\mathfrak{G}_\delta(\omega';X^{\frac{-1}{2}}A_1^{p_1}X^{\frac{-1}{2}},\dots,X^{\frac{-1}{2}}A_n^{p_n}X^{\frac{-1}{2}})\le I$$

where $\widehat{\omega'}=(\frac{w_1}{p_1},\dots,\frac{w_n}{p_n})$ and $\omega'=\frac{\widehat{\omega'}}{\|\widehat{\omega'}\|_1}$. Therefore we have that

$$X\le I \quad \text{implies} \quad \mathfrak{G}_\delta(\omega';A_1^{p_1},\dots,A_n^{p_n})\le X\le I \;\; \text{for } p_1,\dots,p_n\in[1,2]. \tag{2.1}$$

Put $Y=\mathfrak{G}_\delta(\omega';A_1^{p_1},\dots,A_n^{p_n})\le I$. Then by (2.1), we get

$$\mathfrak{G}_\delta(\omega'';A_1^{p_1p_1'},\dots,A_n^{p_np_n'})\le Y\le X\le I$$

for $p_1',\dots,p_n'\in[1,2]$, where $\widehat{\omega''}=(\frac{w_1}{p_1p_1'},\dots,\frac{w_n}{p_np_n'})$ and $\omega''=\frac{\widehat{\omega''}}{\|\widehat{\omega''}\|_1}$. Therefore, by putting $q_i=p_ip_i'$ for $i=1,\dots,n$, we have that

$$X\le I \quad \text{implies} \quad \mathfrak{G}_\delta(\omega'';A_1^{q_1},\dots,A_n^{q_n})\le X\le I \;\; \text{for } q_1,\dots,q_n\in[1,4], \tag{2.2}$$

where $\widehat{\omega''}=(\frac{w_1}{q_1},\dots,\frac{w_n}{q_n})$ and $\omega''=\frac{\widehat{\omega''}}{\|\widehat{\omega''}\|_1}$.

By repeating the same way from (2.1) to (2.2), we have the conclusion. □

Theorem 2.1 also implies generalized Ando-Hiai inequality [9] since $\mathfrak{G}_\delta(\omega;A,B)=A\,\sharp_\alpha\,B$ for $\omega=(1-\alpha,\alpha)$ and $\omega'=\left(\frac{\frac{1-\alpha}{r}}{\frac{1-\alpha}{r}+\frac{\alpha}{s}},\frac{\frac{\alpha}{s}}{\frac{1-\alpha}{r}+\frac{\alpha}{s}}\right)=\left(\frac{(1-\alpha)s}{(1-\alpha)s+\alpha r},\frac{\alpha r}{(1-\alpha)s+\alpha r}\right)$.

Theorem 2.C (Generalized Ando-Hiai inequality [9]). *Let $A, B > 0$. If $A \natural_\alpha B \le I$ for $\alpha \in (0,1)$, then*

$$A^r \natural_{\frac{\alpha r}{(1-\alpha)s+\alpha r}} B^s \le A \natural_\alpha B \le I \quad \text{for } s \ge 1 \text{ and } r \ge 1.$$

The following Theorem 2.2 is a variant from Theorem 2.1.

Theorem 2.2. *Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $\omega = (w_1, \ldots, w_n)$ be a probability vector. For each $i = 1, \ldots, n$ and $q \in \mathbb{R}$, if*

$$\mathfrak{G}_\delta(\omega; A_1^{p_1}, \ldots, A_i^{p_i}, \ldots, A_n^{p_n}) \le A_i^q \quad \text{for } p_1, \ldots, p_n \in \mathbb{R} \text{ with } p_i > q,$$

then

$$\begin{aligned} &\mathfrak{G}_\delta(\omega'; A_1^{p_1}, \ldots, A_{i-1}^{p_{i-1}}, A_i^{p_i'}, A_{i+1}^{p_{i+1}}, \ldots, A_n^{p_n}) \\ &\le \mathfrak{G}_\delta(\omega; A_1^{p_1}, \ldots, A_{i-1}^{p_{i-1}}, A_i^{p_i}, A_{i+1}^{p_{i+1}}, \ldots, A_n^{p_n}) \\ &\le A_i^q \end{aligned}$$

for $p_i' \ge p_i$, where $\widehat{\omega'} = (w_1, \ldots, w_{i-1}, \frac{p_i - q}{p_i' - q} w_i, w_{i+1}, \ldots, w_n)$ and $\omega' = \frac{\widehat{\omega'}}{\|\widehat{\omega'}\|_1}$.

Proof. We may assume $i = 1$ by permutation invariance of $\mathfrak{G}_\delta$.

For $p_1, \ldots, p_n \in \mathbb{R}$ with $p_1 \ge q$, $\mathfrak{G}_\delta(\omega; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \le A_1^q$ if and only if

$$\mathfrak{G}_\delta(\omega; A_1^{p_1 - q}, A_1^{\frac{-q}{2}} A_2^{p_2} A_1^{\frac{-q}{2}}, \ldots, A_1^{\frac{-q}{2}} A_n^{p_n} A_1^{\frac{-q}{2}}) \le I.$$

By applying Theorem 2.1,

$$\begin{aligned} &\mathfrak{G}_\delta(\omega'; A_1^{p_1' - q}, A_1^{\frac{-q}{2}} A_2^{p_2} A_1^{\frac{-q}{2}}, \ldots, A_1^{\frac{-q}{2}} A_n^{p_n} A_1^{\frac{-q}{2}}) \\ &\le \mathfrak{G}_\delta(\omega; A_1^{p_1 - q}, A_1^{\frac{-q}{2}} A_2^{p_2} A_1^{\frac{-q}{2}}, \ldots, A_1^{\frac{-q}{2}} A_n^{p_n} A_1^{\frac{-q}{2}}) \\ &\le I, \end{aligned}$$

holds for $\frac{p_1' - q}{p_1 - q} \ge 1$, where $\widehat{\omega'} = (\frac{p_1 - q}{p_1' - q} w_1, w_2, \ldots, w_n)$. Therefore

$$\mathfrak{G}_\delta(\omega'; A_1^{p_1'}, A_2^{p_2}, \ldots, A_n^{p_n}) \le \mathfrak{G}_\delta(\omega; A_1^{p_1}, A_2^{p_2}, \ldots, A_n^{p_n}) \le A_1^q$$

holds for $p_1' \ge p_1$. $\square$

Next, we show our main result. The following Theorem 2.3 is a generalization of Theorem 1.A, and also a parallel result to (i) $\Longrightarrow$ (iii) in Theorem 1.E.

Theorem 2.3. *Let $A_1, \dots, A_n \in P_m(\mathbb{C})$ and $w_1, \dots, w_n > 0$. If*

$$A_i^{q_i} \geq A_n^{q_n} > 0 \tag{2.3}$$

and

$$\begin{aligned} &\frac{w_1}{p_1 - q_1} \log A_n^{\frac{-q_n}{2}} A_1^{p_1} A_n^{\frac{-q_n}{2}} + \cdots \\ &\quad + \frac{w_{n-1}}{p_{n-1} - q_{n-1}} \log A_n^{\frac{-q_n}{2}} A_{n-1}^{p_{n-1}} A_n^{\frac{-q_n}{2}} + \frac{w_n}{p_n - q_n} \log A_n^{p_n - q_n} \leq 0 \end{aligned} \tag{2.4}$$

hold for $q_i \in \mathbb{R}$, $p_i > q_i$ and $i = 1, \dots, n$, then

$$\mathfrak{G}_\delta(\omega'; A_1^{p'_1}, \dots, A_n^{p'_n}) \leq \mathfrak{G}_\delta(\omega; A_1^{p_1}, \dots, A_n^{p_n}) \leq A_n^{q_n} \quad \text{for all } p'_i \geq p_i \text{ and } i = 1, \dots, n,$$

where $\widehat{\omega} = \left(\frac{w_1}{p_1 - q_1}, \dots, \frac{w_n}{p_n - q_n}\right)$, $\widehat{\omega'} = \left(\frac{w_1}{p'_1 - q_1}, \dots, \frac{w_n}{p'_n - q_n}\right)$, $\omega = \frac{\widehat{\omega}}{\|\widehat{\omega}\|_1}$ and $\omega' = \frac{\widehat{\omega'}}{\|\widehat{\omega'}\|_1}$.

Proof. Applying Theorem 2.B to (2.4), we have

$$\mathfrak{G}_\delta(\omega; A_n^{\frac{-q_n}{2}} A_1^{p_1} A_n^{\frac{-q_n}{2}}, \dots, A_n^{\frac{-q_n}{2}} A_{n-1}^{p_{n-1}} A_n^{\frac{-q_n}{2}}, A_n^{p_n - q_n}) \leq I,$$

so that by (2.3),

$$X_0 \equiv \mathfrak{G}_\delta(\omega; A_1^{p_1}, \dots, A_{n-1}^{p_{n-1}}, A_n^{p_n}) \leq A_n^{q_n} \leq A_1^{q_1}. \tag{2.5}$$

By applying Theorem 2.2 to (2.5) and by (2.3),

$$X_1 \equiv \mathfrak{G}_\delta(\omega_1; A_1^{p'_1}, A_2^{p_2}, \dots, A_n^{p_n}) \leq X_0 \leq A_n^{q_n} \leq A_2^{q_2} \tag{2.6}$$

for $p'_1 \geq p_1$, where $\widehat{\omega}_1 = \left(\frac{w_1}{p'_1 - q_1}, \frac{w_2}{p_2 - q_2}, \dots, \frac{w_n}{p_n - q_n}\right)$ and $\omega_1 = \frac{\widehat{\omega}_1}{\|\widehat{\omega}_1\|_1}$. By applying Theorem 2.2 to (2.6) and by (2.3),

$$X_2 \equiv \mathfrak{G}_\delta(\omega_2; A_1^{p'_1}, A_2^{p'_2}, A_3^{p_3}, \dots, A_n^{p_n}) \leq X_1 \leq X_0 \leq A_n^{q_n} \leq A_3^{q_3}$$

for $p'_1 \geq p_1$ and $p'_2 \geq p_2$, where $\widehat{\omega}_2 = \left(\frac{w_1}{p'_1 - q_1}, \frac{w_2}{p'_2 - q_2}, \frac{w_3}{p_3 - q_3}, \dots, \frac{w_n}{p_n - q_n}\right)$ and $\omega_2 = \frac{\widehat{\omega}_2}{\|\widehat{\omega}_2\|_1}$. By repeating this argument, we can get

$$X_n \equiv \mathfrak{G}_\delta(\omega'; A_1^{p'_1}, \dots, A_n^{p'_n}) \leq X_{n-1} \leq X_0 \leq A_n^{q_n}$$

for $p'_i \geq p_i$ for $i = 1, \dots, n$, where $\widehat{\omega'} = \widehat{\omega}_n = \left(\frac{w_1}{p'_1 - q_1}, \dots, \frac{w_n}{p'_n - q_n}\right)$. □

Remark. (i) in Theorem 1.E, that is, $\log A_1 + \cdots + \log A_n \leq 0$ holds if and only if

$$\frac{1}{p_1} \log A_1^{p_1} + \cdots + \frac{1}{p_n} \log A_n^{p_n} \leq 0 \quad \text{for every } p_i > 0 \text{ and } i = 1, \dots, n.$$

Therefore we recognize that Theorem 2.3 implies (i) $\Longrightarrow$ (iii) in Theorem 1.E by putting $q_1 = \cdots = q_n = 0$ and $w_1 = \cdots = w_n = 1$ since

$$\frac{\frac{1}{p_i}}{\|\widehat{\omega}\|_1} = \frac{\frac{1}{p_i}}{\frac{1}{p_1} + \cdots + \frac{1}{p_n}} = \frac{p_{\neq i}}{\sum_j p_{\neq j}} \quad \text{for } i = 1, \ldots, n$$

ensures $\omega = \frac{\widehat{\omega}}{\|\widehat{\omega}\|_1} = \left(\frac{\frac{1}{p_1}}{\|\widehat{\omega}\|_1}, \ldots, \frac{\frac{1}{p_n}}{\|\widehat{\omega}\|_1}\right) = \left(\frac{p_{\neq 1}}{\sum_i p_{\neq i}}, \ldots, \frac{p_{\neq n}}{\sum_i p_{\neq i}}\right).$

It is well known that we have a variant from Theorem 1.A by replacing A, B with A^q, B^q and p, r with $\frac{p}{q}, \frac{r}{q}$ in Theorem 1.A respectively.

Theorem 2.D ([8]). *Let $A > 0$, $B \geq 0$ and $q > 0$. Then*

$$A^q \geq B^q \quad \textit{implies} \quad A^{-r} \sharp_{\frac{q+r}{p+r}} B^p \leq B^q \leq A^q \quad \textit{for } p \geq q \textit{ and } r \geq 0.$$

Here we show that Theorem 2.3 is a generalization of Furuta inequality via weighted Riemannian mean of n-matrices. Precisely, we show that Theorem 2.3 ensures the following Theorem 2.4 and Theorem 2.4 is a generalization of Theorem 2.D.

Theorem 2.4. *Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $q > 0$. Then $A_i^q \geq A_n^q > 0$ for $i = 1, \ldots, n-1$ implies*

$$\mathfrak{G}_\delta(\omega; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_n^{p_n}) \leq A_n^q \leq A_i^q \tag{2.7}$$

for all $p_i \geq 0$, $i = 1, \ldots, n-1$ and $p_n > q$, where $\widehat{\omega} = \left(\frac{1}{p_1+q}, \ldots, \frac{1}{p_{n-1}+q}, \frac{n-1}{p_n-q}\right)$ and $\omega = \frac{\widehat{\omega}}{\|\widehat{\omega}\|_1}$.

Proof. Assume that $A_i^q \geq A_n^q > 0$ for $q > 0$ and $i = 1, \ldots, n-1$. Then $A_i^q \geq A_n^q > 0$ implies $\log A_i \geq \log A_n$. By (i) $\Longrightarrow$ (iii) in Theorem 1.C, $\log A_i \geq \log A_n$ implies $A_i^{-p_i} \sharp_{\frac{p_i}{q+p_i}} A_n^q \leq I$ for all $p_i \geq 0$. This is equivalent to $A_n^{-q} \sharp_{\frac{q}{q+p_i}} A_i^{p_i} \geq I$, that is, $(A_n^{\frac{q}{2}} A_i^{p_i} A_n^{\frac{q}{2}})^{\frac{q}{p_i+q}} \geq A_n^q$. By taking logarithm, we have $\frac{1}{p_i+q} \log A_n^{\frac{q}{2}} A_i^{p_i} A_n^{\frac{q}{2}} \geq \frac{1}{p_n-q} \log A_n^{p_n-q}$, that is,

$$\frac{1}{p_i+q} \log A_n^{\frac{-q}{2}} (A_i^{-1})^{p_i} A_n^{\frac{-q}{2}} + \frac{1}{p_n-q} \log A_n^{p_n-q} \leq 0 \tag{2.8}$$

for all $p_i \geq 0$, $i = 1, \ldots, n-1$ and $p_n > q$. Summing up (2.8) for $i = 1, \ldots, n-1$, we have

$$\begin{aligned} &\frac{1}{p_1+q} \log A_n^{\frac{-q}{2}} (A_1^{-1})^{p_1} A_n^{\frac{-q}{2}} + \cdots \\ &\quad + \frac{1}{p_{n-1}+q} \log A_n^{\frac{-q}{2}} (A_{n-1}^{-1})^{p_{n-1}} A_n^{\frac{-q}{2}} + \frac{n-1}{p_n-q} \log A_n^{p_n-q} \leq 0. \end{aligned} \tag{2.9}$$

By applying Theorem 2.3 to $(A_i^{-1})^{-q} \geq A_n^q > 0$ and (2.9), we can obtain

$$\mathfrak{G}_\delta(\omega; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_n^{p_n}) \leq A_n^q \leq A_i^q$$

for all $p_i \geq 0 > -q$, $i = 1, \ldots, n-1$ and $p_n > q$. $\square$

Proof of Theorem 2.D. Put $n = 2$, $p_1 = r$ and $p_2 = p$ in Theorem 2.4. Then $\widehat{\omega} = \left(\frac{1}{r+q}, \frac{1}{p-q}\right)$ and $\omega = \left(\frac{p-q}{p+r}, \frac{q+r}{p+r}\right)$. Therefore we obtain the desired result. $\square$

3 3-matrices case

In this section, for the sake of readers' convenience, we state 3-matrices case of Theorems 2.3 and 2.4.

Corollary 3.1. *Let $A, B, C \in P_m(\mathbb{C})$ and $w_1, w_2, w_3 > 0$. If*

$$A^{q_1} \geq C^{q_3} > 0, \quad B^{q_2} \geq C^{q_3} > 0,$$

and

$$\frac{w_1}{p_1 - q_1} \log C^{\frac{-q_3}{2}} A^{p_1} C^{\frac{-q_3}{2}} + \frac{w_2}{p_2 - q_2} \log C^{\frac{-q_3}{2}} B^{p_2} C^{\frac{-q_3}{2}} + \frac{w_3}{p_3 - q_3} \log C^{\frac{-q_3}{2}} C^{p_3} C^{\frac{-q_3}{2}} \leq 0$$

hold for $q_i \in \mathbb{R}$, $p_i > q_i$ and $i = 1, 2, 3$, then

$$\mathfrak{G}_\delta(\omega'; A^{p_1'}, B^{p_2'}, C^{p_3'}) \leq \mathfrak{G}_\delta(\omega; A^{p_1}, B^{p_2}, C^{p_3}) \leq C^{q_3}$$

for all $p_i' \geq p_i$ and $i = 1, 2, 3$, where $\widehat{\omega} = \left(\frac{w_1}{p_1-q_1}, \frac{w_2}{p_2-q_2}, \frac{w_3}{p_3-q_3}\right)$, $\widehat{\omega'} = \left(\frac{w_1}{p_1'-q_1}, \frac{w_2}{p_2'-q_2}, \frac{w_3}{p_3'-q_3}\right)$, $\omega = \frac{\widehat{\omega}}{\|\widehat{\omega}\|_1}$ and $\omega' = \frac{\widehat{\omega'}}{\|\widehat{\omega'}\|_1}$.

Corollary 3.2. *Let $A, B, C \in P_m(\mathbb{C})$ and $q > 0$. Then $A^q \geq C^q > 0$ and $B^q \geq C^q > 0$ implies*

$$\mathfrak{G}_\delta(\omega; A^{-r}, B^{-s}, C^p) \leq C^q \leq A^q \text{ (or } B^q)$$

for $r \geq 0$, $s \geq 0$ and $p > q$, where $\widehat{\omega} = \left(\frac{1}{r+q}, \frac{1}{s+q}, \frac{2}{p-q}\right)$ and $\omega = \frac{\widehat{\omega}}{\|\widehat{\omega}\|_1}$.

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(Masatoshi Ito) Maebashi Institute of Technology, 460-1 Kamisadorimachi, Maebashi, Gunma 371-0816, JAPAN
E-mail address: `m-ito@maebashi-it.ac.jp`